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ON GENERALIZATION OF MENSHOFF'S THEOREM

BY

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ABSTRACT

The condition providing the analyticity for continuous locally univalent functions of complex variables is established. This generalizes the classical Menshoff result on homeomorphic mappings preserving infinitesimal circles.

1. Introduction

The relationship between the basic properties: continuity, monogeneity (existence of a complex derivative) and analyticity of a function of complex variables has been studied in classical and modern function theory by many mathematicians. Interest in questions of this type has increased in connection with constructing a theory of quasiconformal mappings and generalized analytic functions. The aim of the paper is to present a new condition which provides the analyticity of functions.

As is well known, the analytic functions of a complex variable possess various characteristic properties; each of those can be regarded as a definition of analyticity. Such properties are, for example: the monogeneity, conformality of mapping, the conditions of Morera's theorem, uniform approximation by polynomials, etc.

The classical Cauchy–Goursat theorem says (see, e.g., [8]):
If a function $f(z)$ of a complex variable z is continuous and monogenic in a domain $D \subset \mathbb{C}$, then it is analytic in D.

In the terms of real variables, this theorem is formulated as follows:

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A continuous function $f(z) = u(x,y)+iv(x,y)$ is analytic if the functions $u(x,y)$ and $v(x,y)$ are differentiable, and Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

are satisfied everywhere in D.

The monogeneity of $f(z)$, i.e., the existence of the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f'(z), \tag{2}$$

is equivalent to the existence of both the limits

$$\lim_{h\to 0} \left| \frac{f(z+h)-f(z)}{h} \right| \tag{3}$$

(independence of stretching from direction) and

$$\lim_{h\to 0} \operatorname{Arg} \frac{f(z+h)-f(z)}{h} \tag{4}$$

(preserving the angles) at the points where $f'(z) \neq 0$.

Various generalizations and improvements of the Cauchy–Goursat theorem were obtained by Pompeiu, Looman, Montel, Menshoff and other mathematicians. Those rely on weakening the condition of monogeniety. It was natural to find the characterizations of analytic functions either only in terms of stretching at a point or only in terms of preserving the angles.

The first step in this direction was the following theorem of Bohr [1]:

If $w = f(z)$ is a continuous locally univalent mapping of a domain D, for which a finite limit (3) exists and differs from 0 at almost every point of D, then either the function $f(z)$ or the conjugate function $\overline{f(z)}$ is analytic in D.

The next important result is the following theorem of Menshoff [3] based on the second fundamental property of a monogenic function (on preserving the angles).

If a mapping $w = f(z)$ is continuous and locally univalent in a domain D and if at almost every point of D, finite limit (4) exists, then the function $f(z)$ is analytic in D.

Using quasiconformal mappings, Menshoff has obtained in [4] another generalization of the Bohr theorem. Consider a continuous and locally univalent mapping $w = f(z)$ of a domain D of the z-plane onto a domain D^* of the w-plane. For an arbitrary point $z_0 \in D$, take the circle $C(z_0, r) = \{z\colon |z-z_0| = r\} \subset D$, and put

$$H(z_0, r) = \frac{\max_{|z'-z_0|=r} |f(z') - f(z_0)|}{\min_{|z''-z_0|=r} |f(z'') - f(z_0)|}.$$

We say that *the continuous locally univalent function* $f(z)$ *maps the infinitesimal circle* $C(z_0, r)$ *into an infinitesimal circle, if*

$$\lim_{r\to 0} H(z_0, r) = 1.$$

Obviously, the last condition is more general than (3). The important Menshoff's generalization of Bohr's theorem is the following result.

If a function $f(z)$ *is continuous and locally univalent in a domain* D *and maps the infinitesimal circles* $C(z, r)$ *into infinitesimal circles for almost all points* $z \in D$*, then either* $f(z)$ *or* $\overline{f(z)}$ *is analytic in* D*.*

2. Radii of normal neighborhood systems

We shall need the following notation.

Let z be an arbitrary point in $\mathbb{C}$. Assume that some closed neighborhood $\mathcal{G}_t(z)$ of z is defined for any $t \in (0, 1]$. We say that a set of the neighborhoods $\mathcal{G}_t(z)$ of the point z constitutes a **normal system**, if there exists a continuous function v: $\mathbb{C} \to \mathbb{R}$ such that $v(z) = 0$, $v(\zeta) > 0$ for any $\zeta \neq z$. Here $\mathcal{G}_t(z) = \{\zeta \in \mathbb{C}\colon v(\zeta) \leq t\}$ for any $t \in (0, 1]$. Let $\Gamma_t(z) = \{\zeta \in \mathbb{C}\colon v(\zeta) = t\}$ denote the boundary of $\mathcal{G}_t(z)$. The function v is called the **generating function** for a given normal system $\{\mathcal{G}_t(z)\}$ (see, e.g., [6]).

Denote

$$r(z,t) = \inf_{\zeta\in\Gamma_t(z)} |\zeta - z|, \quad \mathcal{R}(z,t) = \sup_{\zeta\in\Gamma_t(z)} |\zeta - z|.$$

These values $r(z, t)$ and $\mathcal{R}(z, t)$ are equal, respectively, to the minimal and the maximal radii of the neighborhood $\mathcal{G}_t(z)$. The limit

$$\Delta(z) = \limsup_{t\to 0} \frac{\mathcal{R}(z,t)}{r(z,t)} \tag{5}$$

is called the **regularity parameter** of the family $\{\mathcal{G}_t(z), 0 < t \leq 1\}$. Any such system $\{\mathcal{G}_t(z)\}$ is called the **regular normal system**, provided $\Delta(z) < \infty$.

Let now f: $D \to D^*$ be a homeomorphism of two bounded domains in $\mathbb{C}$, and let $\{\mathcal{G}_t(z)\}$ be a normal system of neighborhoods of $z \in D$. One can introduce similarly the minimal and the maximal radii for the image of $\mathcal{G}_t(z)$ by

$$r^*(z,t) = \inf_{\zeta\in\Gamma_t(z)} |f(\zeta) - f(z)|, \quad \mathcal{R}^*(z,t) = \sup_{\zeta\in\Gamma_t(z)} |f(\zeta) - f(z)|$$

and

$$\Delta^*(z) = \limsup_{t\to 0} \frac{\mathcal{R}^*(z,t)}{r^*(z,t)}.$$

In these terms Menshoff's theorem says:
If a function $f(z)$ is continuous and locally univalent in a domain D and if $\Delta^*(z) = 1$ at almost every point $z \in D$, then $f(z)$ is analytic in D.

3. Main result

We now present an essential strengthening of the previous classical theorems.

Let p be a real fixed number such that $1 \leq p < \infty$. Put

$$\Theta(z) = \lim_{t\to 0} \frac{mf(B(z, \mathcal{R}(z,t)))}{\pi \mathcal{R}^2(z,t)}, \tag{6}$$

where $B(z,h)$ is the disc $\{\zeta \in \mathbb{C} : |\zeta - z| < h\}$, and mA denotes the Lebesgue two-dimensional measure of a set A.

THEOREM: *If a function $f(z)$ is continuous and locally univalent in a domain D, and for almost every point $z \in D$ there exists a normal regular system of neighborhood $\{\mathcal{G}_t(z)\} \subset D$ such that either the inequality*

$$\limsup_{t\to 0} \frac{\mathcal{R}^*(z,t)}{r(z,t)} \left(\frac{\mathcal{R}(z,t)}{r^*(z,t)}\right)^{p-1} \leq [\Theta(z)]^{(2-p)/2} \tag{7}$$

holds for $1 \leq p \leq 2$ or the inequality

$$\limsup_{t\to 0} \left(\frac{\mathcal{R}^*(z,t)}{r(z,t)}\right)^{p-1} \frac{\mathcal{R}(z,t)}{r^*(z,t)} \leq [\Theta(z)]^{(p-2)/2} \tag{8}$$

holds for $2 \leq p < \infty$, then either $f(z)$ or the conjugate function $\overline{f(z)}$ is analytic in D.

We shall prove the assertion of the theorem for the function $f(z)$ itself, assuming that f is orientation preserving. The proof for $\overline{f(z)}$ is accomplished in a similar way.

The next remark is that for definiteness we can restrict ourselves by the case of inequality (8), i.e., by $2 \leq p < \infty$. The case $1 \leq p < 2$ involving the inequality (7) is treated in a similar way.

We precede the proof of the theorem by several lemmas.

LEMMA 1: *Under the assumption of Theorem, the function $f(z)$ is differentiable almost everywhere in D, and for any Borel set $E \subset D$ we have*

$$\int_E |f'(z)|^2 dxdy < \infty. \tag{9}$$

Proof: Denote

$$k(z) = \limsup_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}\right|.$$

To show that $f(z)$ is differentiable almost everywhere in D, one must verify, in view of Stepanov's theorem [7], that $k(z) < \infty$ almost everywhere in D.

Consider a sequence $\{z_n\}$, $n = 1, 2, \ldots$, of points $z_n \in D$ such that $z_n \to z$ as $n \to \infty$, $z_n \neq z$ for all n, and

$$|f(z_n) - f(z)|/|z_n - z| \to k(z) \quad \text{as } n \to \infty.$$

Let v be the generating function for the system of neighborhood $\mathcal{G}_t = \mathcal{G}_t(z)$. Put $t_n = v(z_n)$ and $w_n = f(z_n)$. Obviously, z_n and w_n are the boundary points of $\mathcal{G}_{t_n}$ and $f(\mathcal{G}_{t_n})$, respectively, and

$$|z_n - z| \geq r(z, t_n), \quad |w_n - w| \leq \mathcal{R}^*(z, t_n).$$

This yields

$$\begin{aligned}\frac{|w_n - w|}{|z_n - z|} &\leq \frac{\mathcal{R}^*(z,t_n)}{r(z,t_n)} \\ &= \left[\left(\frac{\mathcal{R}^*(z,t_n)}{r(z,t_n)}\right)^{p-1}\frac{\mathcal{R}(z,t_n)}{r^*(z,t_n)}\right]^{1/(p-1)}\left[\frac{r^*(z,t_n)}{\mathcal{R}(z,t_n)}\right]^{1/(p-1)}.\end{aligned} \tag{10}$$

The set $\mathcal{G}_{t_n}$ is contained in the disc centered at z with radius $\mathcal{R}(z, t_n)$, while $f(\mathcal{G}_{t_n})$ clearly contains the disc of radius $r^*(z, t_n)$ centered at w. Thus we obtain

$$\left(\frac{r^*(z,t_n)}{\mathcal{R}(z,t_n)}\right)^2 \leq \frac{mf(B(z,\mathcal{R}(z,t_n)))}{mB(z,\mathcal{R}(z,t_n))}.$$

Substitute this bound for the ratio $r^*(z, t_n)/\mathcal{R}(z, t_n)$ into (10) and let n tend to infinity. Therefore the limit of the first factor in the right-hand side of (10) can be estimated by (8), which provides as a result the inequality

$$k(z) \leq \Theta^{1/2}(z). \tag{11}$$

By Stepanov's theorem, f has a total differential at almost all points of D.

To establish (9), observe that the equality (6) can be regarded as the differentiation of the set function $mf(B(z, \mathcal{R}(z, t)))$ over discs. By the Lebesgue theorem (see, e.g., [9], p. 82), this limit exists and is finite almost everywhere in D. Moreover, for every Borel set $E \subset D$, we have

$$\iint_E \Theta(z)\,dx\,dy \leq mf(E).$$

Combining this with inequality (11), one concludes that

$$\iint_E |f'(z)|^2 dxdy \le mf(E) < \infty.$$

The lemma is proved. ∎

Following [8], we call the complex number ξ a **derived number** of the function $f(z)$ at the point z if there is a sequence of numbers $\{\Delta z_n\}$, $\Delta z_n \to 0, n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} \frac{f(z+\Delta z_n) - f(z)}{\Delta z_n} = \xi.$$

Let f be differentiable at a point $z \in D$. Take $\Delta z = \Delta x + i\Delta y = |\Delta z|e^{i\alpha}$ so that $z + \Delta z \in D$. Noting that

$$\Delta f = f(z+\Delta z) - f(z) = f_z \Delta z + f_{\bar z}\Delta \bar z + o(\Delta z),$$

where

$$f_z = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big),$$
$$f_{\bar z} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big),$$

and

$$\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z} = f_z + f_{\bar z}\cdot e^{-2i\alpha},$$

one obtains that the absolute values of the smallest and largest derived numbers of $f(z)$ at z are equal, respectively,

$$l_f = ||f_z| - |f_{\bar z}||, \quad L_f = |f_z| + |f_{\bar z}|.$$

LEMMA 2: *If f is orientation preserving and satisfies the condition of Theorem, then for almost all points $z \in D$ we have equality*

$$\frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} = 1.$$

Proof: Let z_0 be a point of D at which f is differentiable. Fix Δz so that $z_0 + \Delta z \in D$. For simplicity of notations, we can assume that $f(z_0) = z_0 = 0$ as well as

$$\Big|\frac{\Delta f}{\Delta z}\Big| \to L_f \quad \text{and} \quad \Big|\frac{\Delta f}{\Delta \bar z}\Big| \to l_f \quad \text{as } \Delta z \to 0.$$

This can be achieved by a suitable choice of the coordinate axes.

Choose the values $a(t) > 0$ and $b(t) > 0$ so that $a(t)\Delta z \in \Gamma_t(0)$ and $b(t)\Delta \bar z \in \Gamma_t(0)$. Then

$$r(0,t) \le |a(t)\Delta z|, \quad R(0,t) \ge |b(t)\Delta \bar z|,$$

$$r^*(0,t) \leq |f(b(t)\Delta\bar{z})|, \quad R^*(0,t) \geq |f(a(t)\Delta z)|.$$

Combining these inequalities with (8), we obtain

$$\lim_{t\to 0}\Big(\frac{|f(a(t)\Delta z)|}{|a(t)\Delta z|}\Big)^{p-1}\frac{|b(t)\Delta\bar{z}|}{|f(b(t)\Delta\bar{z})|} \leq [\Theta(0)]^{(p-2)/2}. \tag{12}$$

In view of differentiability at 0, the quantity $\Theta(0)$ is equal to the Jacobian of f at 0, and

$$\Theta(0) = J(0,f) = |f_z|^2 - |f_{\bar{z}}|^2 \geq 0.$$

Then (12) yields

$$\frac{|f_z(0)| + |f_{\bar{z}}(0)|}{|f_z(0)| - |f_{\bar{z}}(0)|} = 1. \qquad \blacksquare$$

COROLLARY: *The function $f(z)$ is $\mathbb{C}$-differentiable almost everywhere in D.*

LEMMA 3: *The function f is absolutely continuous on lines (ACL).*

Proof: Let Π be an open rectangle whose sides are parallel to the coordinate axes and let $\overline{\Pi} \subset D$. Denote the vertices of Π by $\xi_k + i\eta_m$, $k, m = 1, 2$. We shall prove that for almost all $y \in (\eta_1, \eta_2)$ the mapping f admits the N-property on the segments $p(y)$ connecting the points $z_1 = \xi_1 + iy$ and $z_2 = \xi_2 + iy$. This means that the image of every set on $p(y)$ of zero measure also has measure zero.

Fix for $z \in D$ a normal regular system $\{\mathcal{G}_t(z)\}$ of neighborhoods of z such that $\mathcal{G}_t(z) \subset D$ for any $t \in (0,1]$. It follows from (8) that for sufficiently small $t > 0$, we have

$$\Big(\frac{\mathcal{R}^*(z,t)}{r(z,t)}\Big)^{p-1}\frac{\mathcal{R}(z,t)}{r^*(z,t)} \leq \Big[\frac{\Psi(B(z,\mathcal{R}(z,t)))}{mB(z,\mathcal{R}(z,t))}\Big]^{(p-2)/2}, \tag{13}$$

where

$$\Psi(A) = mf(A) + \delta(t)mA; \tag{14}$$

here $\delta(t) \to 0$ for $t \to 0$. It is easy to show that

$$\limsup_{t\to 0}\frac{\Psi(B(z,\mathcal{R}(z,t)))}{mB(z,\mathcal{R}(z,t))} = \Theta(z).$$

Put $\Phi(A) = mf(A)$ and denote by I the projection of Π onto the coordinate axis y. Note that $\Pi = p(y) \times I$. Using the set functions Φ and Ψ, we define the functions of open sets $A \subset I$, letting

$$\tilde{\Phi}(A) = \Phi(A \times p(y)), \quad \tilde{\Psi}(A) = \Psi(A \times p(y)).$$

It follows from [5] that for almost all $y \in I$ the values

$$\tilde{\Phi}'(y) = \lim_{h\to 0} \frac{\tilde{\Phi}(B_1(y,h))}{2h}, \quad \tilde{\Psi}'(y) = \lim_{h\to 0} \frac{\tilde{\Psi}(B_1(y,h))}{2h},$$

where $B_1(y,h) = \{z \in I : |\operatorname{Im} z - y| < h\}$, are finite.

Now we suppose that, for some y, the function f does not satisfy the N-property on $p(y)$ and reach a contradiction. In view of our assumption, there is a closed set $E \subset p(y)$ of zero measure for which $H_1(f(E)) > 0$, where H_1 is the one-dimensional Hausdorff measure in $\mathbb{C}$. We shall show that in this case we must have

$$[\tilde{\Psi}'(y)]^{(p-2)/2(p-1)}[\tilde{\Phi}'(y)]^{1/2(p-1)} = \infty.$$

To this end, fix a positive integer k, and let E_k be a set of all $\zeta \in E$ for which $r(\zeta, 1) > 1/k$. Since $E = \bigcup_{k=1}^{\infty} E_k$, $f(E) = \bigcup_{k=1}^{\infty} f(E_k)$, we have

$$H_1(f(E)) \le \sum_{k=1}^{\infty} H_1(f(E_k)).$$

The assumption $H_1(f(E)) > 0$ yields that $H_1(f(E_k)) > 0$ at least for one k. We fix such k and put $\gamma = H_1(f(E_k))$. Then for every system of discs covering $f(E_k)$ the sum of their radii is not less than $\gamma/2$.

Let us divide the low side of $[\xi_1, \xi_2]$ of the rectangle Π (and simultaneously the interval $p(y)$) into $2N$ equal parts, choosing an integer $N > 0$ so that

$$\frac{\xi_2 - \xi_1}{2N} < \frac{1}{k}.$$

Let V_N be the union of all segments in the partition which contains points of E. Since the one-dimensional Lebesgue measure of the set E equals 0 and this set is closed, it follows that $m_1 V_N \to 0$ as $N \to \infty$. For any $\varepsilon > 0$, there exists an integer $N_0(\varepsilon) > 0$ such that

$$m_1 V_N < \varepsilon \tag{15}$$

for any $N \ge N_0$. We fix ε and suitable $N \ge N_0$ for which the inequality (15) holds.

Select on V_N all segments which contain the points of E_k and choose on each E_k one of its points. Denote these points by

$$\zeta_1, \ldots, \zeta_l, \quad 1 \le l \le 2N.$$

Now divide the segment $p(y)$ into $2N(n+1)$ subintervals

$$\sigma_1, \sigma_1, \ldots, \sigma_{2N(n+1)} \tag{16}$$

choosing n so large that it dominates all regularity parameters of the families $\{\mathcal{G}_t(\zeta_i)\}$ (see (5));

$$n > \Delta(\zeta_i), \quad i = 1, \dots, l.$$

Put

$$\rho = \frac{\xi_2 - \xi_1}{2N(n+1)}.$$

For each $i = 1, \dots, l$, there is a value $t_i \in (0,1)$ so that $r(\zeta_i, t_i) = \rho$. The sought value t_i can be found as follows. Let v_i be the generating function of the neighborhood system $\{\mathcal{G}_t(\zeta_i), 0 < t \leq 1\}$. Then the largest value of v_i in $\overline{B}(\zeta_i, \rho)$ is equal to the desired value t_i. Denote $\mathcal{G}_i = \mathcal{G}_{t_i}(\zeta_i)$. The sets $\mathcal{G}_i, i = 1, \dots, l$, cover the original set E_k. Since the disc of radius $\mathcal{R}^*(\zeta_i, t_i)$ centered at $f(\zeta_i)$ contains the set $f(E_m \cap \mathcal{G}_i)$, we obtain

$$\sum_{i=1}^{l} \mathcal{R}^*(\zeta_i, t_i) \geq \gamma/2.$$

Now, following [4], we distribute the segments (16) onto $2(n+1)$ classes, sending the segments

$$\sigma_{2(n+1)j+s}, \quad j = 0, 1, \dots, N-1,$$

into s-class, $s = 1, 2, \dots, 2(n+1)$. Simultaneously, the points $\zeta_1, \dots, \zeta_l$ are also distributed onto these $2(n+1)$ classes.

After such an operation, we obtain that at least for one of these partition classes

$$\sum_{i=1}^{q} \mathcal{R}^*(\zeta_i, t_i) \geq \gamma/4(k+1), \quad q \leq l, \tag{17}$$

and

$$q\rho \leq m_1 V_N < \varepsilon;$$

here $\zeta_1, \dots, \zeta_q$ are all the points of this class.

By the above construction, we have

$$|\zeta_i - \zeta_j| \geq \frac{\xi_2 - \xi_1}{N} - \frac{\xi_2 - \xi_1}{N(n+1)} = \frac{n(\xi_2 - \xi_1)}{N(n+1)}$$

for any $i \neq j, i, j \leq q$. Note that the discs $B(\zeta_i, n\rho)$ are disjoint and $\mathcal{G}_i \subset B(\zeta_i, k\rho)$; hence $\mathcal{G}_i$ are also disjoint. Consider now the rectangle $\Pi_\rho = B_1(y, n\rho) \times p(y)$. Since the set $f(\mathcal{G}_i)$ contains the disc $B(f(\zeta_i), r^*(\zeta_i, t_i))$,

$$\Phi(\Pi_\rho) \geq \pi \sum_{i=1}^{q} r^{*^2}(\zeta_i, t_i). \tag{18}$$

For sufficiently small $t_i > 0$, the inequality (13) implies the relation

$$\mathcal{R}^*(\zeta_i, t_i) \leq$$
$$\pi^{(2-p)/2(p-1)}[\Psi(B(\zeta_i, r(\zeta_i, t_i)))]^{(p-2)/2(p-1)} r^{*^{1/(p-1)}}(\zeta_i, t_i)\mathcal{R}^{-1}(\zeta_i, t_i) r(\zeta_i, t_i)$$

for every $i = 1, \ldots, q$. Summarizing these inequalities and applying Hölder's inequality with the degrees $(p-2)/2(p-1)$, $1/2(p-1)$, and $1/2$, one obtains

$$\sum_{i=1}^{q} \mathcal{R}^*(\zeta_i, t_i) \leq \tag{19}$$
$$\frac{\pi^{(2-p)/2(p-1)}}{n}\left[\sum_{i=1}^{q} \Psi(B(\zeta_i, r(\zeta_i, t_i)))\right]^{(p-2)/2(p-1)} \left[\sum_{i=1}^{q} r^{*^2}(\zeta_i, t_i)\right]^{1/2(p-1)} q^{1/2}.$$

The first factor in the right-hand side of (19) can be estimated applying the relation (14). The second one is estimated by (18). This results in

$$\sum_{i=1}^{q} \mathcal{R}^*(\zeta_i, t_i) \leq \frac{1}{n\sqrt{\pi}}[\Psi(\Pi_\rho)]^{(p-2)/2(p-1)}[\Phi(\Pi_\rho)]^{1/2(p-1)} q^{1/2}.$$

Combining this inequality with (17), we set the low estimate

$$\left[\frac{\Psi(\Pi_\rho)}{2n\rho}\right]^{(p-2)/2(p-1)}\left[\frac{\Phi(\Pi_\rho)}{2n\rho}\right]^{1/2(p-1)} \geq \frac{c}{(q\rho)^{1/2}},$$

with a constant c not depending on N. Taking into account that $q\rho < \varepsilon$ and letting $\rho \to 0$, we obtain

$$[\tilde{\Psi}'(y)]^{(p-2)/2(p-1)}[\tilde{\Phi}'(y)]^{1/2(p-1)} \geq \frac{c}{\varepsilon^{1/2}}. \tag{20}$$

The inequality (20) must hold for arbitrary $\varepsilon > 0$. This yields that for every point $y \in I$, we have the equality

$$[\tilde{\Psi}'(y)]^{(p-2)/2(p-1)}[\tilde{\Phi}'(y)]^{1/2(p-1)} = \infty,$$

which contradicts the fact that the derivatives $\tilde{\Psi}'(y)$ and $\tilde{\Phi}'(y)$ exist and are finite almost everywhere on I.

This contradiction proves that the mapping f admits the N-property on $p(y)$ at almost all points $y \in I$. ∎

Lemma 4: *For a rectangle $\Pi \Subset D$ with the sides parallel to the coordinate axes, we have the equality*

$$\oint_{\partial\Pi} f(z)dz = 0.$$

Proof: By Lemma 1 the derivative $f'(z)$ is integrable over any Borel set $E \Subset D$, i.e. $\iint_E |f'(z)|dxdy < \infty$. This implies that all partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are also integrable. Consider an arbitrary rectangle $\Pi \Subset D$ with the sides parallel to the coordinate axes. Denote its vertices by $\xi_k + i\eta_m$, $k, m = 1, 2$. Then, by Fubini's theorem,

$$\iint_\Pi \frac{\partial u}{\partial x} dxdy = \int_{\eta_1}^{\eta_2} dy \int_{\xi_1}^{\xi_2} \frac{\partial u}{\partial x} dx \quad (u = \operatorname{Re} f(z)),$$

where the inner integral exists for almost all $y \in [\eta_1, \eta_2]$. On the other hand, the function $u(x, y)$ is ACL in Π, i.e. $u(x, y)$ is absolutely continuous with respect to x for almost all $y \in [\eta_1, \eta_2]$. Thus

$$\int_{\xi_1}^{\xi_2} \frac{\partial u}{\partial x} dx = u(\xi_2, y) - u(\xi_1, y).$$

Integrating in $y \in [\eta_1, \eta_2]$, we obtain

$$\int_{\eta_1}^{\eta_2} dy \int_{\xi_1}^{\xi_2} \frac{\partial u}{\partial x} dx = \int_{\eta_1}^{\eta_2} (u(\xi_2, y) - u(\xi_1, y))dy = \oint_{\partial\Pi} udy.$$

Similar equalities are true for the partial derivatives u_y, v_x, v_y. Hence, we have

$$(21) \qquad \begin{aligned} &- \iint_\Pi \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) dxdy + i \iint_\Pi \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) dxdy \\ &= \oint_{\partial\Pi} udx - vdy + i \oint_{\partial\Pi} vdx + udy = \oint_{\partial\Pi} f(z)dz. \end{aligned}$$

Now recall that the function $f(z)$ is $\mathbb{C}$-differentiable almost everywhere in Π and hence equations (1) are satisfied almost everywhere in Π. Therefore, both terms in the left-hand side of (21) are equal to 0, which yields

$$\oint_{\partial\Pi} f(z)dz = 0. \qquad \blacksquare$$

Now the proof of Theorem is completed in the following way. By Lemmas 1 and 2 the function $f(z)$ is $\mathbb{C}$-differentiable in D and has there the derivative

$f'(z)$ which is integrable over D. In addition, f satisfies by Lemma 3 the N-property on almost all lines parallel to the coordinate axes. Since, by Lemma 4,

$$\oint_{\partial\Pi} f(z)dz = 0$$

for an arbitrary rectangle Π, it follows from the classical Morera's theorem that $f(z)$ is complex analytic on the domain D.

Remark: The inequalities (7) and (8) admit generalization to wider classes of the mappings (see [2]).

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